ANALYTICAL MECHANICS
Lagrangian Dynamics

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Agenda

1. Introduction

Lagrangian dynamics – What is it all about?

Lagrangian dynamics...

- is a fundamental approach within **analytical mechanics**
- provides one of the most powerful tools used in modelling and simulations: **the Lagrange equations of motion**
- does not require to analyze **forces on isolated parts** of a mechanical system
- treats a **mechanical system as a whole**, based on scalar quantities defining the entire system

2. Analytical Description of Mechanical Systems

Basic generalized quantities

- The number of degrees of freedom: s

- Generalized coordinates

$$q_1, q_2, \ldots, q_s$$

completely specify **configuration of a system** in the physical three-dimensional space

- Generalized velocities

$$\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_s$$

specify how the generalized coordinates **change in time**

From the Newtonian to analytical viewpoint

- Transformation equations:

$$\mathbf{r}_i = \mathbf{r}_i(t, q_1, \ldots, q_s)$$
$$\dot{\mathbf{r}}_i = \dot{\mathbf{r}}_i(t, q_1, \ldots, q_s, \dot{q}_1, \ldots, \dot{q}_s)$$

  where $i = 1, 2, \ldots, n_b$

- From the Cartesian space to the "generalized space":

$$\left.\begin{array}{c} x_1,\ y_1,\ z_1 \\ x_2,\ y_2,\ z_2 \\ \ldots \\ x_{n_b},\ y_{n_b},\ z_{n_b} \end{array}\right\} \quad \rightarrow \quad q_1, q_2, \ldots, q_s$$

  Usually

$$s < 3n_b \qquad \text{(for 3D systems)}$$
$$s < 2n_b \qquad \text{(for 2D systems)}$$

Motion from the analytical viewpoint

Let t_c denote a selected (current) instant of time during motion of a given mechanical system.

- Generalized coordinates $q_i(t_c)$ describe current **configuration** of the system (positions of its members)
- Generalized velocities $\dot{q}_i(t_c)$ specify current **tendency** of the system towards change in q_i
- To determine future behaviour of the system (for $t > t_c$) we have to know **both** the generalized coordinates and velocities at t_c

Motion from the analytical viewpoint

- **State** of a system ("state of motion") – values of all the variables describing the system (in space and time) at some instant t_c

  State = Current configuration + Tendency

  i.e. the state is defined by $q_1, q_2, \ldots, q_s$ and $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_s$.

- **Evolution** of a system – changes of the state of the system in time

Motion from the analytical viewpoint

[Thornton & Marion, 2004]

- **Configuration space** – s-dimensional space:

$$\{q_1, q_2, \ldots, q_s\}$$

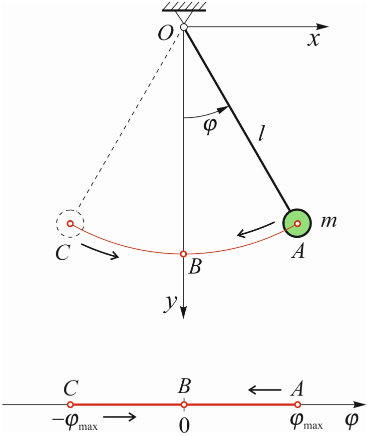
- Motion of a mechanical system may be interpreted as motion of a point

$$P = (q_1, q_2, \ldots, q_s)$$

in the configuration space along a curve which is called a **configuration path** or **configuration trajectory**

- Each point $P(t)$ on the path represents the **configuration** of the entire system at some given time instant t

Example: Cartesian space vs. Configuration space



Motion from the analytical point of view [Thornton & Marion, 2004]

- **Phase space** (or **state space**) – $2s$-dimensional space:

$$\{q_1, q_2, \ldots, q_s, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_s\}$$

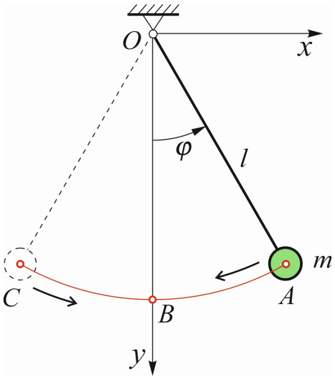
- Evolution of a given system may be interpreted as motion of a point

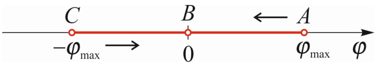
$$X = (q_1, \ldots, q_s, \dot{q}_1, \ldots, \dot{q}_s)$$

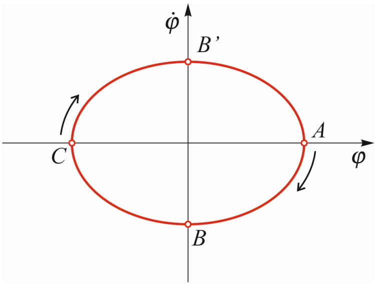
  in the phase space along a curve which is called a **phase trajectory**

- Each point $X(t)$ on the phase trajectory represents the **state** of the entire system at some given time instant t

Example: Cartesian space vs. State space







3. Lagrange Equations

Dynamics problem statement

Suppose that the state of a mechanical system is given at initial time t_0:

$$\begin{aligned} q_1(t_0) &= \bar{q}_1\,, & \dot{q}_1(t_0) &= \bar{\dot{q}}_1 \\ q_2(t_0) &= \bar{q}_2\,, & \dot{q}_2(t_0) &= \bar{\dot{q}}_2 \\ &\dots & &\dots \\ q_s(t_0) &= \bar{q}_s\,, & \dot{q}_s(t_0) &= \bar{\dot{q}}_s \end{aligned}$$

How to determine the motion of the system for $t_0 < t \leq t_1$?

Mechanical energy of a system

Assumptions:

1. the system is subjected to **geometric** and **ideal** constraints
2. all the active forces are **potential** (conservative)

A mechanical system can be characterized by the two **scalar quantities**:

- **Kinetic energy**

$$T = T(t, q_1, \ldots, q_s, \dot{q}_1, \ldots, \dot{q}_s)$$

- **Potential energy**

$$V = V(t, q_1, \ldots, q_s)$$

The Lagrange function [Hand & Finch, 1998]

- **Total mechanical energy** of the system:

$$E = T + V$$

- The **Lagrange function (Lagrangian)** for the system:

$$L = T - V$$

$$L = L(t, q_1, \ldots, q_s, \dot{q}_1, \ldots, \dot{q}_s)$$

The Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0\,, \qquad i = 1, 2, \ldots, s$$

- The Lagrange equations for a mechanical system form a set of s **second order ordinary differential equations** (ODEs) with $2s$ unknown functions $q_1, \ldots, q_s, \dot{q}_1, \ldots, \dot{q}_s$
- To solve the equations for an arbitrary time interval $t_0 < t \leq t_1$, the initial conditions must be known:

$$\begin{array}{cc} q_1(t_0) = \overline{q}_1\,, & \dot{q}_1(t_0) = \overline{\dot{q}}_1 \\ q_2(t_0) = \overline{q}_2\,, & \dot{q}_2(t_0) = \overline{\dot{q}}_2 \\ \ldots & \ldots \\ q_s(t_0) = \overline{q}_s\,, & \dot{q}_s(t_0) = \overline{\dot{q}}_s \end{array}$$

The Lagrange equations – Alternative forms

If some **additional external forces** act on the system (e.g. due to excitation or dissipation), they can be introduced to the dynamic equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i^{np}\,, \qquad i = 1, 2, \ldots, s$$

where Q_i^{np} – a generalized **non-potential force**.

The Lagrange equations – Alternative forms

If V is independent on the generalized velocities $\dot{q}_j$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i^{np} \qquad i = 1, 2, \ldots, s$$

or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i^{p} + Q_i^{np} \qquad i = 1, 2, \ldots, s$$

where Q_i^p – a generalized **potential force**:

$$Q_i^p = -\frac{\partial V}{\partial q_i}$$

Lagrangian dynamics – Remarks

- In order to describe a mechanical system, a set of generalized coordinates may be chosen **in many different ways**
- The resulting equations of motion have more or less complex form
- There are **no general rules** for selecting the "most suitable" generalized coordinates
- It can be proven that the Newtonian and Lagrangian equations are **entirely equivalent**
- If all constraints are ideal, within the Lagrangian formulation there is no need to take into account passive forces (constraints forces)

Lagrangian vs. Newtonian dynamics [Thornton & Marion, 2004]

The Newtonian approach...

- focuses on the forces acting **on** a body
- treats systems in terms of **vector** quantities, e.g. force, velocity, angular momentum, torque
- associates a certain **cause** (force) with a definite **effect** (motion)

The Lagrangian approach...

- deals with quantities (energies) associated **with** the body
- describes systems by **scalar** operations in configuration space
- associates motion with the attempt of nature to achieve a certain **purpose**: to minimize the action integral

PROCEDURE

for deriving equations of motion of planar mechanical systems

1. Specify the **number of particles** and **rigid bodies** within the mechanical system ($n_b = n_p + n_r$)
2. Specify the **number of degrees of freedom** s of the system
3. Choose a proper set of **generalized coordinates**: $q_1, q_2, \ldots, q_s$
4. Specify a fixed Cartesian coordinate system xy
5. Write the **transformation equations** for all members ($i = 1, 2, \ldots, n_b$), i.e. express positions of particles and mass centers of rigid bodies:

$$x_i = x_i(t, q_1, \ldots, q_s), \qquad y_i = y_i(t, q_1, \ldots, q_s)$$

6. Derive the **transformation equations** related to velocities ($i = 1, 2, \ldots, n_b$):

$$\dot{x}_i = \dot{x}_i(t, q_1, \ldots, q_s, \dot{q}_1, \ldots, \dot{q}_s), \quad \dot{y}_i = \dot{y}_i(t, q_1, \ldots, q_s, \dot{q}_1, \ldots, \dot{q}_s)$$

7. Write the **kinetic energy** of the system, using **König's theorem** for rigid bodies:

$$T = \frac{1}{2}\sum_{i=1}^{n_b}\left[m_i(\dot{x}_i^2 + \dot{y}_i^2) + I_i\omega_i^2\right]$$

(Note that for particles the term $I_i\omega_i^2$ vanishes)

8. Write the **potential energy** of the system:

$$V = V_g + V_s$$

9 Combine T and V by writing the **Lagrangian**: $L = T - V$

10 Specify the **non-potential forces** acting on the system (if any); derive the **generalized forces** Q_i^{np} corresponding to particular generalized coordinates q_i

11 Compute all the necessary **derivatives** of L:

$$\frac{\partial L}{\partial \dot{q}_i}, \quad \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right), \quad \frac{\partial L}{\partial q_i}, \qquad i = 1, 2, \ldots, s$$

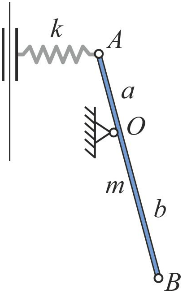
12 Finally, write the **equations of motion** for the system according to the **Lagrange equations**:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = Q_i^{np}, \qquad i = 1, 2, \ldots, s$$

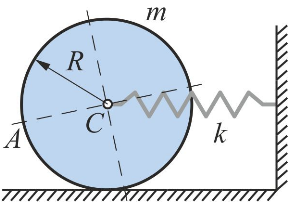
4. Problems

Derive equations of motion for the given mechanical systems

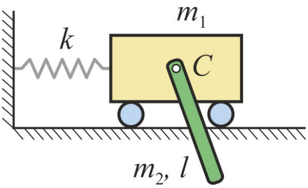
A homogeneous rod AB of mass m and length $l = a + b$ can rotate around the horizontal axis which passes through the point O. A linear spring of constant k is attached to the end A. The spring is mounted in such a way that it remains horizontal during motion of the system. The spring is unstretched when the rod is in the vertical position. Assume that $|OA| = a = l/3$.

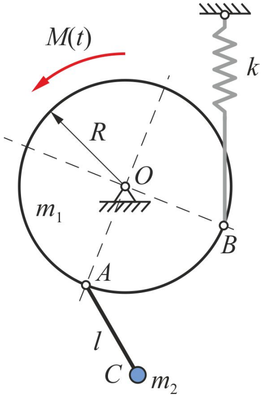


A homogeneous disc of mass m and radius R can roll without slipping on a ground surface. The body is attached to a wall via a linear spring of constant k, mounted at the center C. The spring remains horizontal during motion of the system, and is unstretched as the radius AC is oriented horizontally.



A truck of mass m_1 can move on a ground surface without friction. The body is attached to a wall via a linear spring of constant k. At the mass centre C, a homogeneous bar of mass m_2 and length l is attached via a rotational joint.





A homogeneous drum of mass m_1 and radius R can rotate around the horizontal axis which passes through the point O. A particle of mass m_2 is mounted to the disc via a joint A and a massless bar of length l. Moreover, a linear spring of constant k is attached to the point B of the disc. The spring is unstretched as the radius OB is oriented horizontally. Motion of the system is excited by an external moment

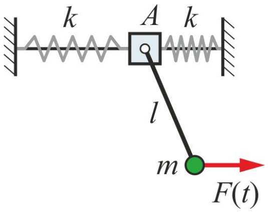
$$M(t) = M_0 \sin(\Omega t)$$

Consider small vibrations of the system, thus assume that the spring remains vertical.

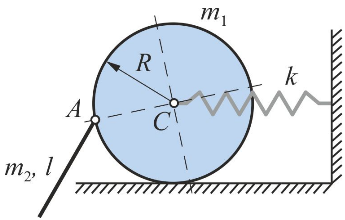
A massless homogeneous rod of length l is attached to a light slider by a joint A. A small body of mass m is mounted to the other end of the rod. Horizontal motion of the slider is restricted by two springs of constant k. Vibrations of the system are driven by an external periodic force

$$F(t) = F_0 \sin(\Omega t)$$

which is applied horizontally to the particle.



A homogeneous disc of mass m_1 and radius R can roll without slipping on a ground surface. The body is attached to a wall via a linear spring of constant k, mounted at the center C. The spring is unstretched as the radius CA is oriented horizontally. A homogeneous rod of mass m_2 and length l is attached at the joint A.



5. Summary

Conclusions and final remarks

- Within the Lagrangian dynamics, equations of motion for a given system can be derived based on **scalar quantities** describing the **entire system**
- The Lagrange equations provide a general rule for **unified treatment** of degrees of freedom which have different nature (translations and rotations)
- The Lagrangian formulation of mechanics is **entirely equivalent** to the Newtonian approach
- The Lagrangian dynamics **does not** constitute a new theory: the only novelty is the method used to obtain equations of motion

Bibliography

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Hand L.N., Finch J.D., *Analytical Mechanics*. Cambridge University Press, 1998.

Thornton S.T., Marion J.B., *Classical Dynamics of Particles and Systems*. Brooks/Cole, 2004.